Lecture 4-Spherical Mirror Resonators
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10:36 AM

Reading for Feb 25: Section 11.1

Resonator Input and Output

If $R_1 \neq 1$ and/or $R_2 \neq 1$, then a fraction of the cavity light will be transmitted through the mirrors. This transmitted light will have exactly the same temporal and spatial characteristics as the light in the cavity.

Each time the pulse of light in our cavity reflects from one of its partially transmitted mirror, a weak replica of that pulse will emerge from the mirror and leave the cavity. The spectrum of this emitted light must be identical to the spectrum in the cavity.

Moreover, the transmission through the mirrors does not depend on whether the light is leaving the cavity or entering it. Thus the transmittance into the cavity shows the same spectral characteristics as the mode distribution inside the cavity.

If we direct a $\delta$-function pulse of light toward the left mirror of our cavity and tune it to arrive at that mirror at $t=0$, then a fraction of that pulse will be transmitted into the cavity and find itself at $t=0$ at $t=0$. It will then bounce back and forth in the cavity exactly as we have previously discussed for a slightly glossy Fabry-Perot cavity.

The arriving $\delta$-function pulse has a white-light spectrum and the light in the cavity ends up with a spectrum that is highly peaked around discrete multiples of $\frac{\lambda}{k} = \frac{c}{2d}$ (the free spectral range of the cavity). That same highly-peaked spectrum emerges from the cavity on the right. Thus a Fabry-Perot cavity or "Etalon" has a high transmittance for frequencies within $\Delta \nu$ of its discrete resonances. If you send white light at it, it will pick out and transmit only frequencies near those resonances.

But what happens to the rejected light? It is reflected from the left mirror. That reflected light is no longer white; it has spectral holes cut into it at the cavity resonances because of interferences between the directly reflected wave and the transmitted fraction of the circulating wave.
A Fabry-Perot etalon is a high resolution spectrometer. Solid Fabry-Perot etalons make excellent frequency control elements in laser systems.

For example, what are the spectral characteristics of a solid glass F-P etalon that is 0.5 cm thick and has mirrored coatings \( R_1 = R_2 = 0.85 \)?

\[
\begin{align*}
\eta &\approx 1.5 \text{ for glass} \quad r = 0.85 \\
\nu_f &= \frac{c}{2d} \approx \frac{2 \times 10^{10} \text{ cm/s}}{1.0 \text{ cm}} \approx 20 \text{ GHz} \\
\nu &= \frac{\pi \sqrt{r}}{1-r} \approx 20 \\
\delta \nu &= \frac{\nu_f}{\nu} \approx 1 \text{ GHz} \\
I_{\text{min}} &= I_{\text{max}} \cdot \frac{1}{1 + (2F/\pi)^2} \approx I_{\text{max}} \cdot 0.006
\end{align*}
\]

So this etalon has spectral peaks every 20 GHz, 1 GHz wide, and the contrast between transmission on and off resonance is about 0.006 to 1. Pretty good!

A Fabry-Perot's peaks can be tuned by moving a mirror to change \( d \). Such scannable etalons are used in optical spectrum analysis and to tune laser sources.

**Mode Densities in 2D and 3D resonators**

Before we turn our attention to resonators with spherical mirrors, let's take a moment to generalize planar mirror cavities to 2- and 3-dimensional. Our purpose is not so much about actual using such resonators. Rather, it is about learning how to calculate mode densities. Mode densities will prove important later in this course when we examine lasers.

For a 1-D planar-mirror cavity (a Fabry-Perot etalon), the mode density \( N(\nu) \) is:
\[ M(y) = \frac{y}{c}. \]

To generalize to 2- and 3-dimensions, let's start by considering a perfectly reflective (perfectly conducting) square pipe:

![Diagram of a square pipe]

The Helmholtz equation \( \nabla^2 U(r) + k^2 U(r) = 0 \) has solutions with constant amplitude and phase along \( \mathbb{R} \) (i.e., 1-D solutions):

\[ U(x, y, z) = A Y(y) Z(z) \]

where

\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_x^2 \right) Y(y) Z(z) + k^2 Y(y) Z(z) = 0 \]

Multiplying on the left by \( \frac{1}{Y(y) Z(z)} \)

\[ \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} + \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} + k^2 = 0 \]

or

\[ \left( \frac{1}{Z(z)} \frac{\partial^2 Z(z)}{\partial z^2} + k_x^2 \right) + \left( \frac{1}{Y(y)} \frac{\partial^2 Y(y)}{\partial y^2} + k_y^2 \right) = 0 \]

Because these two parenthesized expressions are functions of different variables, they must each vanish independently.

\[ \frac{\partial^2 Z(z)}{\partial z^2} + k_x^2 Z(z) = 0 \]
\[ \frac{\partial^2 Y(y)}{\partial y^2} + k_y^2 Y(y) = 0 \]

We have already solved these differential equations for the same boundary conditions \( \partial_z (z = 0) = Z(z = 0) = 0 \) and \( Y(y = 0) = Y(y = d) = 0 \).
\[ Z_1(z) = \sin k_z z \quad , \quad k_z = \frac{m\pi}{d} \quad \text{where} \quad m = 1, 2, 3 \ldots \]
\[ Y_1(y) = \sin k_y y \quad , \quad k_y = \frac{l\pi}{d} \quad \text{where} \quad l = 1, 2, 3 \ldots \]

So:
\[ U(x, y, z) = A \sin k_y y \sin k_z z \]
\[ = A \sin \frac{b\pi y}{d} \sin \frac{m\pi z}{d} \]

and
\[ k = \frac{2\pi}{\lambda} = \frac{2\pi c}{c} = \sqrt{k_y^2 + k_z^2} \]

\[ Y = \frac{c}{2\pi} \sqrt{\left(\frac{b\pi}{d}\right)^2 + \left(\frac{m\pi}{d}\right)^2} = \frac{c}{2d} \sqrt{b^2 + m^2} \]

The allowed frequency modes of this 2-D cavity are sinusoidal along both the \( \hat{x} \) and \( \hat{z} \) axes, with \( 0, 1, 2 \ldots \) nodal lines parallel to those axes:

\[ l=1 \quad m=1 \]
\[ l=1 \quad m=2 \]
\[ l=2 \quad m=3 \]

The 3-D case can be solved in the same fashion. Now we have a cubic resonator:

\[ \mathbf{e} = \mathbf{y} \]

\[ U(x, y, z) = A \sin k_x x \sin k_y y \sin k_z z \]

\[ k_x^2 = \frac{\omega^2}{c^2} + k_y^2 + k_z^2 \]
\[ k_x = \frac{a\pi}{d} \quad b = 1, 2, 3 \ldots \]
\[ k_y = \frac{b\pi}{d} \quad l = 1, 2, 3 \ldots \]
\[ k_z = \frac{m\pi}{d} \quad m = 1, 2, 3 \ldots \]

\[ \omega(x, y, z) = A \sin \frac{a\pi x}{d} \sin \frac{b\pi y}{d} \sin \frac{m\pi z}{d} \]
\[ U(x, y, z) = A \sin \frac{\pi y}{d} \sin \frac{2\pi z}{d} \sin \frac{\pi x}{d} \]
\[ \nu = \frac{c}{2d} \sqrt{\nu_0^2 + k^2 + k_0^2} \]

Let's determine the mode density for the 3-D case:

Although mode density is defined in terms of frequency interval, we'll find it useful to find the number of modes per wavenumber interval and then convert that value into modes per frequency interval.

In wavenumber space:

![Wavenumber space diagram](image)

there is one wave mode for every cube of size: \( \frac{\pi}{d} \times \frac{\pi}{d} \times \frac{\pi}{d} \) or equivalently of volume \( \left( \frac{\pi}{d} \right)^3 \). With two linearly independent polarizations there are actually \( 2 \) light modes in that volume.

How many modes are contained in a sphere of radius \( k = \sqrt{k_x^2 + k_y^2 + k_z^2} \)?

The volume of that sphere is \( \frac{4}{3} \pi k^3 \) but there are modes only in the \( k_x > 0, k_y > 0, k_z > 0 \) octant. We can determine the number of modes in the sphere by dividing the volume of that octant by the volume per mode:

\[ N(k) = \frac{\frac{4}{3} \pi k^3}{\frac{\pi}{d}^3} = \frac{k^3 d^3}{\pi^2} \]

now we convert to frequency: \( k = \frac{2\pi \nu}{c} \)

\[ N(\nu) = \frac{8}{3} \pi \frac{\nu^3 d^3}{c^3} \]

is the number of light modes in a sphere of radius \( \nu \).

But we want to know how many modes \( 8N(\nu) \) are in
\[ \delta N(\nu) = \frac{\delta N(\nu)}{\delta \nu} \]
\[ \delta N(\nu) = \frac{8\pi}{c^2} \frac{\nu^2}{\delta \nu} \]

To find the mode density (modes per frequency per volume), we further divide by the volume of the cavity, \( d^3 \):
\[ M(\nu) = \frac{8\pi\nu^2}{c^3} \]

**Spherical Mirror Resonators**

The stationary mode solutions in a resonator are built from traveling waves that traverse the cavity over and over again, each time returning to the same spatial structure and phases.

In our plane mirror cavities, those stationary modes have planar wavefronts that are unchanged by reflection from a cavity mirror and phase that are invariant (up to an integer multiple of \( 2\pi \)) following one roundtrip of the cavity.

These two criteria, that the wavefronts be unchanged by reflection from a cavity mirror and that the phase be invariant after a roundtrip, must be satisfied for the stationary solutions of any resonator, including one with spherical mirrors.

The first criterion will be satisfied if the wavefronts are parallel to the mirror surface.

The second criterion will be satisfied everywhere if the on-axis phase is invariant after a roundtrip.

For a resonator consisting of spherical mirrors, the stationary modes will have spherical wavefronts. Gaussian (and Hermite-Gaussian or Legendre-Gaussian) beams satisfy this requirement in the paraxial approximation. So, we can expect Gaussian beam solutions to a spherical mirror cavity.

Given a two-mirror cavity with mirrors having radii of curvature \( R_1, R_2 \) separated by distance \( d \), we know the properties of the Gaussian beam. That
beam's wavefronts must match the mirror curvatures at \( z = z_0 \) (mirror \( M_1 \)) and \( z = z_1 + d \) (mirror \( M_2 \)).

However, we have to be careful about the sign conventions for the mirrors and the Gaussian beam. For mirror, concave surfaces have \( R < 0 \). But the curvature of the Gaussian beam is \( R > 0 \) to the left of \( z = 0 \) and \( R > 0 \) to the right.

As drawn, \( R_1 = R_{\text{beam}}(z_0) < 0 \)
\[ R_2 = -R_{\text{beam}}(z_1 + d) < 0 \]

But if \( M_1 \) is convex:

now, \( R_1 = R_{\text{beam}}(z_1) > 0 \)
\[ R_2 = -R_{\text{beam}}(z_1 + d) < 0 \]

But what are \( z_1 \) and \( z_0 \) for a given pair of mirrors and their separation \( d \)?

In homework exercise 3.1-5, solutions are given for these values in terms of the Gaussian beam's curvature at points \( z = z_1 \) and \( z = z_1 + d \). If we set the beam curvatures at these points equal to the mirror curvatures:

\( R_{\text{beam}}(z_1) = R_1 \) and \( R_{\text{beam}}(z_1 + d) = -R_2 \), then:

\[ z_1 = \frac{-d(R_2 + d)}{R_2 + R_1 + 2d} \]
\[
\sigma_z = \frac{d (R_1 + d)(R_2 + d)((R_2 + R_1 + d))}{(R_2 + R_1 + 2d)^2}
\]

For a Gaussian beam, \( \sigma_z \) must be real. So that will only be a Gaussian beam mode in the cavity if:

\[
d (R_1 + d)(R_2 + d)(R_2 + R_1 + d) \leq 0
\]

we can divide both sides by \( R_1^2, R_2^2 \) \((R_1^2, R_2^2 > 0)\):

\[
(1 + \frac{d}{R_2}) (1 + \frac{d}{R_1}) (\frac{d}{R_2} + \frac{d}{R_1} + \frac{d^2}{R_1 R_2}) \leq 0
\]

\[
(1 + \frac{d}{R_2}) (1 + \frac{d}{R_1}) (\frac{d}{R_2} - 1) \leq 0
\]

\[
0 \leq (1 + \frac{d}{R_2}) (1 + \frac{d}{R_1}) \leq 1
\]

For symmetric resonators, \( R_1 = R_2 \equiv -R \) (where \( R \geq 0 \))

In this case, \( \sigma_z = -\frac{d}{2} \), \( \sigma_z + d = \frac{d}{2} \)

The resonator is stable if:

\[
0 \leq \left(1 - \frac{d}{R}\right)^2 \leq 1
\]

\[
0 \leq \frac{d}{R} \leq 2
\]
Three special cases:
\[ \frac{d}{R} = 0 \] is our old friend the Fabry-Perot etalon.
\[ \frac{d}{R} = 1 \] is the confocal resonator, smallest beams on the mirrors: \[ d = R = 2z_0 \] is "confocal parameter".
\[ \frac{d}{R} = 2 \] is the concentric resonator, smallest beam waist.

These three are just barely stable.

So far, we have found Gaussian beam solutions to spherical mirror cavities when those solutions exist, but now we need to find the stable longitudinal modes. We next try to find beams that have on-axis phase invariance after a round trip of the cavity.

The on-axis phases at the two mirrors are:
\[ \phi(z_1) = k_0 z_1 - \frac{\pi}{2} \]
\[ \phi(z_1 + d) = k_0 (z_1 + d) - \frac{\pi}{2} \]
So \[ \Delta \phi \] for half a round trip is:
\[ \Delta \phi = k_0 d - \Delta \frac{\pi}{2} \]
where \[ \Delta \frac{\pi}{2} = \phi(z_1 + d) - \phi(z_1) \]
for an invariant phase after a round trip,
\[ 2\phi = 2k_0 d - 2\Delta \frac{\pi}{2} = 2\pi q_0 \quad q_0 = 1, 2, 3 \ldots \]

\[ k_0 = \frac{2\pi q_0 + 2\Delta \frac{\pi}{2}}{2d} \]

Since \[ k_0 = \frac{2\pi c}{\lambda} \]
\[ \lambda = \frac{c}{2d} \left( q_0 + \frac{\Delta \frac{\pi}{2}}{\pi} \right) q_0 = 1, 2, 3 \ldots \]

so the spectrum resembles the Fabry-Perot spectrum, but with an offset.
\[ \nu = \nu_f \left( q + \frac{\Delta \gamma}{\pi} \right) \]

For the Hermite-Gaussian beams, that axial phase factor is multiplied by the factor \(1 + l + m\):

\[ \nu = \nu_f \left( q + (1 + l + m) \frac{\Delta \gamma}{\pi} \right) \]

So the different \( l \) and \( m \) modes have different resonant frequencies. \( l \) and \( m \) identify the transverse mode while \( q \) identifies the longitudinal mode.

Homework, due Feb. 25

10.1-4
10.2-2
10.2-3
10.2-6
10.3-1