Longitudinal coherence.

Our previous examination of coherence along the propagation direction looked at it as a temporal issue. We considered a single point in space and looked for a relationship between the wave at two different times.

This time, let's do the reverse: let's consider a single moment in time and look for a relationship between two different points in space along the propagation direction. Doing that will illustrate that temporal coherence is actually a property of the wave that measures the length along the propagation direction over which there is a coherent relationship and negligible fluctuations.

Let's consider a wave with perfect transverse (spatial) coherence: a plane wave. But let's allow that plane wave to be partially coherent along the propagation direction:

\[ U(\vec{r}, t) = a(t - z/c) e^{i2\pi\nu(t - z/c)} \]

where \( a(t - z/c) \) is a complex function that affects the wave along its propagation direction. This description of the wave requires that any fluctuations in the wave come from the source itself, and not from any scattering or absorption along the path. In other words, once produced, the wave propagates forward without additional changes in amplitude for any constant value of \( z - ct \).

The mutual coherence function for the wave at two points \( \vec{r}_1 \) and \( \vec{r}_2 \) is

\[ G(\vec{r}_1, \vec{r}_2, \tau) = \lim_{T \to \infty} \frac{1}{T} \int_{-\infty}^{\infty} a^*(t - \frac{z_1}{c}) e^{i2\pi\nu(t - \frac{z_1}{c} + \tau)} a(t - \frac{z_2}{c} + \tau) e^{i2\pi\nu(t - \frac{z_2}{c} + \tau)} dt \]

\[ = \lim_{T \to \infty} \frac{1}{T} e^{i2\pi\nu(\tau - \frac{z_2 - z_1}{c})} \int_{-\infty}^{\infty} a^*(t - \frac{z_1}{c}) a(t - \frac{z_2}{c} + \tau) dt \]
\[ t' = t - z_1/c, \quad \tau = t + z_2/c \]

\begin{align*}
\ &= e^{j2\pi \nu_0 (\tau - \frac{z_2}{c})} \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} a^*(t) a(t' - \frac{z_2}{c}) dt' \\
\ &= G_a (\tau - \frac{z_2}{c}) e^{j2\pi \nu_0 (\tau - \frac{z_2}{c})}
\end{align*}

where \( G_a (\tau) = \langle a(t) a(t+\tau) \rangle \) is the autocorrelation function of \( a \).

We can define a normalized version of \( G \):

\[ g (\tau_1, \tau_2) = j_G \left( \tau - \frac{z_2}{c} \right) e^{j2\pi \nu_0 (\tau - \frac{z_2}{c})} \]

where \( j_G (\tau) = \frac{G_a (\tau)}{G_a (0)} \)

For \( z_1 = z_2 \), we recover the temporal coherence function.

For this modified plane wave:

\[ g (\tau_1, \tau_2) = j_G (\tau) e^{j2\pi \nu_0 \tau} \]

For \( z_1 \neq z_2 \) and \( \tau = 0 \), we get:

\[ g (\tau_1, \tau_2) = j_G \left( -\frac{z_2 - z_1}{c} \right) e^{j2\pi \nu_0 \left( -\frac{z_2 - z_1}{c} \right)} \]

There is clearly a perfect mapping between the wave's temporal coherence properties at time delay \( \tau \) and the wave's longitudinal coherence at spatial separation \( (z_2 - z_1) = c\tau \).

If \( |z_2 - z_1| < cT_c \), then the wave is coherent between these two points.

If \( |z_2 - z_1| > cT_c \), then there is no clear relationship between the wave at these two points.

We can think of the wave as consisting of unrelated pulses of length \( cT_c \) and duration \( T_c \).

A similar examination of spherical waves would yield a comparable result.
Interference

Let's consider two waves that are passing through a single point in space \( \vec{r} \) at time \( t \).

The instantaneous intensity at \( \vec{r} \) is

\[
I(t) = |U_1(\vec{r},t) + U_2(\vec{r},t)|^2
\]

and the time average intensity at that point is:

\[
I_{\text{tot}} = \langle I(t) \rangle = \langle |U_1 + U_2|^2 \rangle
\]

\[
= \langle |U_1|^2 \rangle + \langle |U_2|^2 \rangle + \langle U_1^* U_2 + U_2^* U_1 \rangle
\]

\[
= I_1 + I_2 + 2 \Re \langle U_1^* U_2 \rangle
\]

\[
= I_1 + I_2 + 2 \Re \langle G_{12} \rangle
\]

where

\[
G_{12} = \langle U_1^* U_2 \rangle
\]

is the cross-correlation of \( U_1 \) and \( U_2 \).

All of the interference is due to the last "interference" term, which we can write:

\[
2 \sqrt{I_1 I_2} \Re \langle G_{12} \rangle
\]

where \( G_{12} \) is the normalized cross-correlation.

We can write \( \Re \langle G_{12} \rangle \) in terms of an amplitude and phase as:

\[
\Re \langle G_{12} \rangle = |g_{12}| \cos \Theta
\]

where

\[
\Theta = \tan^{-1} \left[ \frac{\Im \langle G_{12} \rangle}{\Re \langle G_{12} \rangle} \right]
\]

\( \Theta \) depends on the relative phases of \( U_1 \) and \( U_2 \).

It's easy to see that:

\[
I_{\text{min}} = I_1 + I_2 - 2 \sqrt{I_1 I_2} |g_{12}|
\]

\[
I_{\text{max}} = I_1 + I_2 + 2 \sqrt{I_1 I_2} |g_{12}|
\]
If we continuously vary the relative phase between \( u_1 \) and \( u_2 \), \( I_{\text{tot}} \) will vary periodically:

\[
I_{\text{tot}} = I_{\text{min}} \leq I_{\text{tot}} \leq I_{\text{max}}
\]

and we can define a contrast or interference visibility:

\[
\frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} = \frac{2|I_1 I_2| |g_{12}|}{I_1 + I_2}
\]

If \( I_1 = I_2 \), the contrast is simply the normalized cross-correlation \( |g_{12}| \).

Now suppose \( u_2 \) is a time-delayed copy of \( u_1 \):

\[
u_2(t) = u_1(t - \tau)
\]

we could obtain such a delayed copy using the reflections from a transparent plate or a Michelson interferometer.

Then \( g_{12} = G(\tau) \), \( g_{12} = g(\tau) \), and \( I_1 = I_2 \), so:

\[
I_{\text{tot}} = 2 I_1 (1 + |g(\tau)| \cos \Phi(\tau))
\]

and the contrast is \( |g(\tau)| \).

For \( \tau < \tau_c \), \( g(\tau) \approx 1 \) and the interference fringes have essential perfect contrast.

For \( \tau > \tau_c \), \( g(\tau) \approx 0 \) and there is no interference.

If we gradually scan \( \tau \), we obtain an interferogram:

\[
I_{\text{tot}} = 2 I_1 (1 + |g(\tau)|)
\]
Fourier Transform Spectroscopy

The total intensity can be written in terms of $G(\tau)$:

$$I_{tot} = 2I_1 + 2 \text{Re}(G(\tau))$$

$$= 2I_1 + G^*(\tau) + G(\tau)$$

If we subtract $2I_1$ from $I_{tot}$ and Fourier transform, we obtain the power spectrum in frequency space:

$$\mathcal{F}(G^*(\tau) + G(\tau)) = 2S(\nu)$$

So an interferogram can give us $g(\tau)$, $G(\tau)$, and even $S(\nu)$ -- the power spectrum of the wave.

Obtaining a spectrum from an autocorrelation is known as Fourier Transform Spectroscopy. FTS can be used to characterize any light wave, whether it comes directly from a source or has passed through a medium, so it can characterize a source or a medium or both.

To study an absorbing medium, for example, we can set up a light source, a sample of the medium, and a Michelson Interferometer.

We scan the mirror more to characterize the light.
we scan the mirror once to characterize the light source itself:

\[ \text{scan the mirror} \rightarrow \mathcal{F} \rightarrow \nu \]

Then we insert the sample and scan the mirror again to characterize the source and sample:

\[ \text{scan the mirror with sample} \rightarrow \mathcal{F} \rightarrow \nu \]

Subtracting the second spectrum from the first yields the absorption spectrum of the sample:

\[ \mathcal{S}(\nu) - \mathcal{S}'(\nu) \]

An incoherent light source can be used to generate the light—a bulb or glowing bar. The frequency resolution is inversely proportional to the scan duration.

The scanning of the mirror must be fine enough to resolve the fringes, making the technique easiest to use in the infrared, where it is known as FTIR (Fourier Transform Infrared spectroscopy).

Photons

So far, we have treated light as a classical wave that follows from Maxwell's equations. But light also has quantum characteristics that we'll need to consider in order to understand how light interacts with matter.

In the quantum theory of light, light is quantized into packets known as photons. The energy and momentum of a light mode can only change by an integer multiple of the photon energy and momentum:

\[ E_\gamma = \hbar \omega \quad \text{where} \quad \hbar = \frac{\hbar}{2\pi} = 1.06 \times 10^{-34} \text{J.s} \]

\[ P_\gamma = \hbar k \]

Our approach to developing this quantum theory of light...
our approach to developing this quantum theory of light will be to draw the analogy between the photons in a particular light mode and the excitations of a particular harmonic oscillator.

Our first step will be to recast the classical harmonic oscillator so that we have a direct analogy between its harmonic motion and a light wave. In particular, we'll define light analogs for position and momentum in the harmonic oscillator.

Our second step will be to advance to the quantum harmonic oscillator and then use our analogy to obtain a quantum theory of light.
The Classical Harmonic Oscillator (1.2)

\[ \frac{1}{2} \kappa \mathbf{v}^2 + \frac{1}{2} m \dddot{x} \]

The classical harmonic oscillator consists of a mass \( m \) on a spring with spring constant \( \kappa \). Newton’s second law (\( F = ma = m \frac{\ddot{x}}{dt^2} \)) for the mass is:

\[ \frac{\ddot{x}}{dt^2} + \omega^2 x = 0 \quad ; \quad \omega = \sqrt{\frac{\kappa}{m}} \]

The general solution to this differential equation is:

\[ x(t) = A \cos(\omega t + \phi) \]

where \( A \) and \( \phi \) can be any positive, real values.

We can rewrite the cosine term:

\[ x(t) = A \mathbf{Re}(e^{i(\omega t + \phi)}) \]

\[ = A \mathbf{Re}(e^{i\phi} e^{i\omega t}) \]

\[ = A \mathbf{Re}\left( \frac{A_0}{A} (A e^{i\phi}) e^{-i\omega t} \right) \]

\[ = A_0 \mathbf{Re}(a e^{-i\omega t}) \]

where \( A_0 \) is real and \( a \) is a dimensionless complex number:

\[ a = \frac{A}{A_0} e^{i\phi} \]

The momentum of the oscillator (\( m \dot{v} = m \frac{d\dot{x}}{dt} \)) is:

\[ p(t) = -m \omega A \sin(\omega t + \phi) = m \omega \mathbf{Im}(A e^{i\phi} e^{-i\omega t}) \]

\[ = m \omega A_0 \mathbf{Im}(a e^{-i\omega t}) \]

And the oscillator’s energy:

\[ E = \frac{p^2}{2m} + \frac{1}{2} \kappa x^2 \]

and since \( \kappa = m \omega^2 \),
\[ E = \frac{m\omega^2}{2} A_0^2 \left[ \text{Im}(ae^{-jwt}) \right]^2 + \frac{m\omega^2}{2} A_0^2 \left[ \text{Re}(ae^{-jwt}) \right] \]

But \( (\text{Im}(z))^2 + (\text{Re}(z))^2 = |z|^2 \), so

\[ E = \frac{m\omega^2}{2} A_0^2 |a|^2 \]

where \( E_0 = \frac{m\omega^2}{2} A_0^2 \)

we can also write \( E \) as:

\[ E = E_0 (\hat{x}^2 + \hat{p}^2) \]

where \( \hat{x} \) and \( \hat{p} \) are dimensionless:

\[ \hat{x} = \frac{x}{A_0} = \text{Re}(ae^{-jwt}) \]

\[ \hat{p} = \frac{p}{m\omega A_0} = \text{Im}(ae^{-jwt}) \]

So the energy of the oscillator, in units of \( E_0 \), is given by the square modulus of \( ae^{-jwt} \). \( \hat{x} \) and \( \hat{p} \) are the two "quadratures" of the motion -- they represent orthogonal directions in the complex plane and we must "add them in quadrature" (sum their squares) to obtain \( E \).

The two quadratures are scaled versions of the position and momentum:

\[ x = A_0 \hat{x} = \sqrt{2E_0} \hat{x} \quad ; \quad \hat{x} = \frac{m\omega}{2E_0} x \]

\[ p = m\omega A_0 \hat{p} = \sqrt{2mE_0} \hat{p} \quad ; \quad \hat{p} = \frac{1}{\sqrt{2E_0}} p \]

Note: \( |a| \) may not be 1!
Quantizing the harmonic oscillator

To find the stationary states of the harmonic oscillator, we must solve the time-independent Schrödinger equation:

\[-\frac{\hbar^2}{2m} \frac{d^2 \Psi(x)}{dx^2} + \omega^2 x^2 \frac{\Psi(x)}{2} = E \Psi(x)\]

The solutions are:

\[\Psi_n(x) = C_n H_n\left(x\sqrt{\frac{\hbar \omega}{m}}\right) \exp\left(-\frac{x^2 \hbar \omega}{2m}\right)\]

\[E_n = \hbar \omega \left(\frac{1}{2} + n\right) \quad n = 0, 1, 2, \ldots\]

where \(\hbar = \frac{h}{2\pi}\), \(H_n(x)\) are the Hermite polynomials, and \(C_n\) is a normalization constant.

In a given state \(\Psi_n\), the probability of finding the oscillator at \(x\) is \(|\Psi(x)|^2\) and \(C_n\) is chosen so that:

\[\int_{-\infty}^{\infty} |\Psi_n|^2 \, dx = 1\]

Each stationary state has its own phase evolution:

\[\Psi_n(x, t) = \Psi_n(x) \exp\left(-i E_n t / \hbar\right)\]

but its probability of being found at \(x\) is

\[|\Psi_n(x, t)|^2 = |\Psi_n(x)|^2\]

and clearly has no time dependence. As a result, these stationary states have no dynamics. However, more general solutions to the time-dependent Schrödinger equation change with time and are
Schrödinger equation change with time and are composed of superpositions of the stationary states. The general solutions are:

$$\Psi(x,t) = \sum_n c_n \psi_n(x) e^{-i E_n t / \hbar}$$

and the probability of finding it at $x$ is:

$$|\Psi(x,t)|^2 = \sum_n c_n^* c_n \psi_n^* \psi_n e^{i (E_n - E_n) t / \hbar}$$

and is time-dependent. The probability distribution changes with time. It is possible to form a wave packet that is more localized in $X$, but only at the expense of additional uncertainty in energy and momentum. The system behaves more classically with respect to position, as the wavepacket bounces back and forth in the potential well at frequency $\nu = \omega / 2\pi$.

We can also write the quantum solution using the dimensionless quadratures that we developed for the classical solution. But we must choose a value for $A_0$ (or equivalently for $E_0$). Let's choose $E_0 = \hbar \omega$. Then:

$$\hat{X} = \frac{m \omega}{\hbar k \omega m} \hat{x} = \sqrt{\frac{m \omega}{2 \hbar}} \hat{x}$$

$$\psi_n(x) = c_n \psi_n(\sqrt{2} \hat{X}) e^{-\frac{x^2}{2}}$$

$$E_n = \hbar \omega \left( n + \frac{1}{2} \right)$$

Now $|\psi(x,t)|^2$ gives the probability that the oscillator is in a particular quadrature $\hat{x}$.
The analogy to light

The wave equation for electromagnetic waves is

$$\nabla^2 \mathcal{E}(\mathbf{r}, t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathcal{E}(\mathbf{r}, t) = 0$$

Let's consider a plane wave solution

$$\mathcal{E}(\mathbf{r}, t) = e^{j \mathbf{k} \cdot \mathbf{r} - j \omega t}$$

The wave equation becomes:

$$-\nabla^2 \mathcal{E}(t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \mathcal{E}(t) = 0$$

Since $\omega = k \cdot c$, this becomes:

$$\frac{\partial^2}{\partial t^2} \mathcal{E}(t) + \omega^2 \mathcal{E}(t) = 0$$

That is precisely the equation of a harmonic oscillator, with the electric field playing the role of the oscillator's position.

The solution is:

$$\mathcal{E}(t) = A_0 e^{-j \omega t}$$

where $A_0$ is positive and real, and $\omega$ is complex and dimensionless.

The energy in the wave mode is...
\[ E = \frac{1}{2} \varepsilon \int_V \mathbf{E}^*(\mathbf{r}, t) \mathbf{E}(\mathbf{r}, t) \, dV \]

where \( \varepsilon \) is the electric permittivity and \( V \) is the volume containing the wave (can be arbitrarily large ... we'll resolve any issues shortly).

Since \( \mathbf{E}(\mathbf{r}, t) = e^{i \mathbf{k} \cdot \mathbf{r}} \mathbf{E}(t) \), the integral over volume is simple:

\[ E = \frac{1}{2} \varepsilon V \mid \mathbf{E}(t) \mid^2 \]

\[ = \frac{1}{2} \varepsilon V A_0^2 \mid a \mid^2 \]

Defining \( E_0 = \frac{1}{2} \varepsilon A_0^2 V \), we get

\[ E = E_0 \mid a \mid^2 \]

This is so analogous to the harmonic oscillator, that we will set our energy scale \( E_0 = \hbar \omega \). That choice simply selects \( A_0 \) for us: \( A_0^2 = \frac{2 \hbar \omega}{\varepsilon V} \).

\[ E = \hbar \omega \mid a \mid^2 \]

\[ \mathbf{E}(t) = \sqrt{\frac{2 \hbar \omega}{\varepsilon V}} \left[ a \, e^{-i \omega t} \right] \]

In analogy to the harmonic oscillator, we can define quadrature variables \( \hat{x} \) and \( \hat{p} \) for the field:

\[ E = E_0 \mid a \mid^2 = \hbar \omega \mid a \mid^2 = \hbar \omega (\hat{x}^2 + \hat{p}^2) \]

where \( \hat{x} = \text{Re}(ae^{-i \omega t}) \), \( \hat{p} = \text{Im}(ae^{-i \omega t}) \)

To quantize the field, we use the analogy and expect stationary states only for

\[ E_n = \hbar \omega \left( \frac{1}{2} + n \right) \quad n = 0, 1, 2, \ldots \]

only certain discrete energies are allowed for a given natural mode and those energies are
a given optical mode and those energies are spread how apart—the energy of a photon of that mode.

**Next time (March 17):**

- 50-minute open-book midterm
- Questions similar style to homework questions

Then we will continue with photons